

ASPECTS OF STABLE POLYNOMIALS

STEVE FISK

This article is an introduction to the properties of stable polynomials in several variables with real or complex coefficients. These polynomials are defined in terms of where the polynomial is non-vanishing.

Definition 1. $\mathcal{H}_d(\mathbb{C}) = \left\{ \begin{array}{l} \text{All polynomials } f(x_1, \dots, x_d) \text{ with complex coefficients} \\ \text{such that } f(\sigma_1, \dots, \sigma_d) \neq 0 \text{ for all } \sigma_1, \dots, \sigma_d \\ \text{in the right half plane. If we don't need to specify } d \\ \text{we simply write } \mathcal{H}(\mathbb{C}). \end{array} \right.$

We call such polynomials *stable polynomials*. In one variable they are often called Hurwitz stable.

This article is a companion to [7] where we studied polynomials non-vanishing in the upper half plane.

Definition 2. $\mathcal{U}_d(\mathbb{C}) = \left\{ \begin{array}{l} \text{All polynomials } f(x_1, \dots, x_d) \text{ with complex coefficients} \\ \text{such that } f(\sigma_1, \dots, \sigma_d) \neq 0 \text{ for all } \sigma_1, \dots, \sigma_d \\ \text{in the upper half plane. If we don't need to specify } d \\ \text{we simply write } \mathcal{U}(\mathbb{C}). \end{array} \right.$

Such polynomials are called *upper polynomials*.

Since multiplication by \mathfrak{z} takes the right half plane to the upper half plane, we have the important fact that

$$f(x_1, \dots, x_d) \in \mathcal{U}_d(\mathbb{C}) \iff f(\mathfrak{z}x_1, \dots, \mathfrak{z}x_d) \in \mathcal{H}_d(\mathbb{C})$$

Consequently, many of the properties of $\mathcal{H}(\mathbb{C})$ follow immediately from the properties of $\mathcal{U}(\mathbb{C})$, and their proofs will be omitted. See [1–7, 13] for more information about $\mathcal{U}(\mathbb{C})$ and $\mathcal{H}(\mathbb{C})$. We use the notation in [7], and let RHP denote the right half plane.

We do not cover well-known topics in one variable such as Routh-Hurwitz, the Edge theorem, and Kharitonov theory. See [11, 12].

1. COMPLEX COEFFICIENTS

These results are proved following the corresponding arguments for upper polynomials. The main difference is the disappearance of minus signs for $\mathcal{H}(\mathbb{C})$.

Fact 1.

- (1) Suppose $f(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C})$.
- (a) If $\alpha \neq 0$ then $\alpha f(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C})$.

Date: March 3, 2008.

- (b) If $a_1 > 0, \dots, a_d > 0$ then $f(a_1x_1, \dots, a_dx_d) \in \mathcal{H}_d(\mathbb{C})$.
- (c) If $\Re(\sigma_1) > 0, \dots, \Re(\sigma_d) > 0$ then $f(x_1 + \sigma_1, \dots, x_d + \sigma_d) \in \mathcal{H}_d(\mathbb{C})$.
- (d) If $\Re(\sigma) > 0$ then $f(\sigma, x_2, \dots, x_d) \in \mathcal{H}(\mathbb{C})$.
- (e) $f(x_1 + y, x_2, \dots, x_d) \in \mathcal{H}(\mathbb{C})$.
- (f) $f(x, x, x_3, \dots, x_d) \in \mathcal{H}(\mathbb{C})$.
- (2) If $f(\mathbf{x}) \in \mathcal{H}(\mathbb{C})$ then $f(\mathbf{ia}, x_2, \dots, x_d) \in \mathcal{H}(\mathbb{C}) \cup \{0\}$ if $a \in \mathbb{R}$.
- (3) $\mathcal{H}(\mathbb{C})$ is closed under multiplication and extracting factors.
- (4) If $\sum_0^n f_i(\mathbf{x}) y^i \in \mathcal{H}(\mathbb{C})$ then $\sum_0^n f_i(\mathbf{x}) y^{n-i} \in \mathcal{H}(\mathbb{C})$.
- (5) If $f(\mathbf{x}) \in \mathcal{H}(\mathbb{C})$ then $\partial_{x_i} f(\mathbf{x}) \in \mathcal{H}(\mathbb{C}) \cup \{0\}$.
- (6) If $\sum f_i(\mathbf{x}) y^i \in \mathcal{H}(\mathbb{C})$ then all coefficients $f_i(\mathbf{x})$ are in $\mathcal{H}(\mathbb{C}) \cup \{0\}$.

We have interlacing for both $\mathcal{H}(\mathbb{C})$ and $\mathcal{U}(\mathbb{C})$; later we will see there is another kind of interlacing when all the coefficients are real. Note that interlacing is symmetric for stable polynomials.

Definition 3.

$$\begin{array}{ll} f(\mathbf{x}) \xleftarrow{H} g(\mathbf{x}) & \text{if and only if} \quad f(\mathbf{x}) + y g(\mathbf{x}) \in \mathcal{H}(\mathbb{C}) \\ f(\mathbf{x}) \xleftarrow{U} g(\mathbf{x}) & \text{if and only if} \quad f(\mathbf{x}) + y g(\mathbf{x}) \in \mathcal{U}(\mathbb{C}) \end{array}$$

Fact 2.

- (1) If $\sum_0^n f_i(\mathbf{x}) y^i \in \mathcal{H}(\mathbb{C})$ then $f_i(\mathbf{x}) \xleftarrow{H} f_{i+1}(\mathbf{x})$ for $i = 0, \dots, n-1$, provided f_i and f_{i+1} are not both zero.
- (2) If $f \in \mathcal{H}(\mathbb{C})$ then $f \xleftarrow{H} \partial_{x_i} f(\mathbf{x})$.
- (3) Suppose $f, g \in \mathcal{H}(\mathbb{C})$
 - (a) $fg \xleftarrow{H} gh$ iff $h \in \mathcal{H}(\mathbb{C})$ and $f \xleftarrow{H} g$.
 - (b) If $f \xleftarrow{H} g$ then $g \xleftarrow{H} f$.
 - (c) If $f \xleftarrow{H} g$ and $f \xleftarrow{H} h$ then $f \xleftarrow{H} g + h$.
 - (d) If $f \xleftarrow{H} g$ and $h \xleftarrow{H} g$ then $f + h \xleftarrow{H} g$.
 - (e) If $f \xleftarrow{H} g \xleftarrow{H} h$ then $f + h \xleftarrow{H} g$.
- (4) (Hermite-Biehler) Suppose that $f(\mathbf{x})$ is a polynomial, and write $f(\mathbf{x}) = f_e(\mathbf{x}) + f_o(\mathbf{x})$ where $f_e(\mathbf{x})$ (resp. $f_o(\mathbf{x})$) consists of all terms of even (resp. odd) degree. Then

$$f \in \mathcal{H}(\mathbb{C}) \text{ if and only if } f_e \xleftarrow{H} f_o.$$

Definition 4. If $f(\mathbf{x})$ is a polynomial then $f^H(\mathbf{x})$ is the sum of all terms with highest total degree.

Fact 3. Suppose that $f(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C})$ is a stable polynomial of degree n .

- (1) $f^H(\mathbf{x})$ is a stable homogeneous polynomial.
- (2) f^H is the limit of stable homogeneous polynomials such that all monomials of degree n have non-zero coefficient.
- (3) All the coefficients of f^H have the same argument.

2. REAL COEFFICIENTS

Definition 5. \mathcal{H}_d is the subset of $\mathcal{H}_d(\mathbb{C})$ whose coefficients are all positive.

The restriction that the coefficients are all positive is natural:

Fact 4. *If $f(\mathbf{x}) \in \mathcal{H}(\mathbb{C})$ has all real coefficients then all coefficients have the same sign.*

Proof. Suppose $f(\mathbf{x})$ has degree n . We can write $f(\mathbf{x})$ as a limit of $f_\epsilon(\mathbf{x})$ where f_ϵ is stable, and all coefficients of terms of degree at most n are non-zero. Since the signs of a stable polynomial in one variable are all the same, an easy induction shows that all coefficients of f_ϵ have the same sign. Taking limits finishes the proof. \square

Fact 5. *If a, b, c, d are positive then $a + bx + cy + dxy \in \mathcal{H}_2$.*

Proof. If $f(x, y) = a + bx + cy + dxy = 0$ where $x = r + \mathbf{i}s$ then

$$\Re(y) = -\frac{(ac + bcr + adr + bdr^2 + bds^2)}{(c + dr)^2 + d^2s^2}$$

If $r + \mathbf{i}s \in \text{RHP}$ then $r > 0$ so $\Re(y) \notin \text{RHP}$. Thus $f(x, y)$ can not have both x, y in the right half plane, so f is stable. \square

There are constructions of polynomials in \mathcal{H} and $\mathcal{H}(\mathbb{C})$ using determinants, and they follow from the corresponding results for \mathbf{U} .

Fact 6. *Suppose that A is skew-symmetric, S is symmetric, and all D_i are positive definite.*

$$\begin{aligned} |I + x_1 D_1 + \cdots + x_d D_d + \mathbf{i}S| &\in \mathcal{H}_d(\mathbb{C}) \\ |I + x D_1 + \cdots + x_d D_d + A| &\in \mathcal{H}_d \end{aligned}$$

Next we introduce polynomials that share the properties of both stable and upper polynomials.

Definition 6. Let $P_d^+ = \mathcal{H}_d \cap \mathbf{U}_d$. Interlacing in P_d^+ is defined as usual

$$f \xleftarrow{P} g \text{ iff } f + yg \in P_d^+$$

and is equivalent to $f \xleftarrow{H} g$ and $f \xleftarrow{U} g$

In one variable, P_1^+ consists of all polynomials with all real roots, no positive roots, and all positive coefficients. Here's an example of a polynomial with positive coefficients that is not in P_2^+ . Let $f(x, y) = x(x + 3) + y(x + 1)(x + 2)$. The roots of $f(x, 1 + \mathbf{i})$ lie in the second and third quadrants, so $f(x, 1 + \mathbf{i}) \notin \mathbf{U}_2(\mathbb{C})$, and hence is not in P_2^+ .

3. ANALYTIC CLOSURE

Definition 7. $\widehat{\mathcal{H}}_d(\mathbb{C})$ is the uniform closure on compact subsets of $\mathcal{H}_d(\mathbb{C})$.

Fact 7.

- (1) $e^{\mathbf{x} \cdot \mathbf{y}} \in \widehat{\mathcal{H}}_{2d}(\mathbb{C})$ and $e^{\mathbf{x} \cdot \mathbf{x}} \in \widehat{\mathcal{H}}_d(\mathbb{C})$.
- (2) If $f(\mathbf{x}, \mathbf{y}) \in \mathcal{H}_{2d}(\mathbb{C})$ then $e^{\partial_{\mathbf{x}} \cdot \partial_{\mathbf{y}}} f(\mathbf{x}, \mathbf{y}) \in \mathcal{H}_{2d}(\mathbb{C})$.
- (3) If $f(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C})$ and $g(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C})$ then $f(\partial_{\mathbf{x}})g(\mathbf{x}) \in \mathcal{H}_d(\mathbb{C}) \cup \{0\}$.
This also holds for $f \in \widehat{\mathcal{H}}_d(\mathbb{C})$.
- (4) Suppose T is a non-trivial linear transformation defined on polynomials in d variables. $T(e^{\mathbf{x} \cdot \mathbf{y}}) \in \widehat{\mathcal{H}}_{2d}(\mathbb{C})$ if and only if T maps $\mathcal{H}_d(\mathbb{C}) \cup \{0\}$ to itself.

Example 1. The Hadamard product of $f(y)$ and $g(\mathbf{x}, y)$ is given by

$$f * g = \left(\sum a_i y^i \right) * \left(\sum g_i(\mathbf{x}) y^i \right) = \sum a_i g_i(\mathbf{x}) y^i$$

- (1) It is easy to verify that $f(x) \in P_1^+$ then $f(xy) \in \mathcal{H}_2(\mathbb{C})$.
- (2) The linear transformation $\exp: x^i \mapsto x^i/i!$ determines a map $\mathcal{H}_1(\mathbb{C}) \longrightarrow \mathcal{H}_1(\mathbb{C})$. This follows from the above since the generating function $f(x, y)$ of \exp is $g(xy)$ where $g = \sum x^i/(i!i!) \in P_1^+$.
- (3) The linear transformation $g \mapsto f * g$ has generating function

$$\sum (f(y) * y^i \mathbf{x}^{\mathbf{I}}) \frac{v^i \mathbf{u}^{\mathbf{I}}}{i! \mathbf{I}!} = e^{\mathbf{x} \cdot \mathbf{u}} \exp f(yv)$$

This is in $\widehat{\mathcal{H}}(\mathbb{C})$, so we have a map $P_1^+ \times \mathcal{H}_d(\mathbb{C}) \longrightarrow \mathcal{H}_d(\mathbb{C})$. It is surprising [8] that the Hadamard also maps $\mathcal{H}_1 \times \mathcal{H}_1 \longrightarrow \mathcal{H}_1$.

Fact 8. Suppose that $f(\mathbf{x}, \mathbf{y})$ is a polynomial, and define $T(g) = f(\mathbf{x}, \partial_{\mathbf{y}})g$. The following are equivalent.

- (1) $T: \mathcal{H}_d(\mathbb{C}) \longrightarrow \mathcal{H}_d(\mathbb{C}) \cup 0$.
- (2) $f(\mathbf{x}, \mathbf{y}) \in \mathcal{H}_{2d}(\mathbb{C})$.

4. POSITIVE INTERLACING

If $f \xleftarrow{H} g$ then $f + \sigma g$ is stable for all σ in the right half plane. If we restrict ourselves to just the real numbers in the right half plane we get *positive interlacing*.

Definition 8.

$$\begin{array}{ll} f(\mathbf{x}) \stackrel{P}{\sim} g(\mathbf{x}) & \text{if and only if} \quad f(\mathbf{x}) + r g(\mathbf{x}) \in P^+ \quad \text{for all } r > 0 \\ f(\mathbf{x}) \stackrel{H}{\sim} g(\mathbf{x}) & \text{if and only if} \quad f(\mathbf{x}) + r g(\mathbf{x}) \in \mathcal{H} \quad \text{for all } r > 0 \end{array}$$

Clearly both $\stackrel{H}{\sim}$ and $\stackrel{P}{\sim}$ are reflexive. In one variable $f \stackrel{P}{\sim} g$ is equivalent to common interlacing. That is, $f \stackrel{P}{\sim} g$ if and only if there is an $h \in P_1^+$ so that $h \stackrel{P}{\leftarrow} f$ and $h \stackrel{P}{\leftarrow} g$. In general it follows from Fact 2 that

Fact 9.

- (1) If $f, g \in P^+$ and there is an h such that $h \xleftarrow{P} f$ and $h \xleftarrow{P} g$ then $f \stackrel{P}{\sim} g$.
- (2) If $f \in P^+$ then $\partial_{x_i}(f) \stackrel{P}{\sim} \partial_{x_j}(f)$.

Here are some elementary properties of $\stackrel{H}{\sim}$ and $\stackrel{P}{\sim}$.

Fact 10. Suppose r, s are positive.

- (1) $f \stackrel{H}{\sim} g$ iff $rf \stackrel{H}{\sim} sg$.
- (2) $f \stackrel{H}{\sim} g$ iff $f + rg \stackrel{H}{\sim} g$
- (3) If $h \in \mathcal{H}$ then $f \stackrel{H}{\sim} g$ iff $fh \stackrel{H}{\sim} hg$.
- (4) $f \stackrel{H}{\sim} g$ iff $g \stackrel{H}{\sim} f$.
- (5) $f \stackrel{P}{\sim} g \implies f \stackrel{H}{\sim} g$
- (6) $f \stackrel{H}{\sim} g \not\implies f \stackrel{P}{\sim} g$
- (7) $f \xleftarrow{P} g \implies f \stackrel{P}{\sim} g$
- (8) $f \stackrel{P}{\sim} g \not\implies f \xleftarrow{P} g$
- (9) $f \xleftarrow{H} g \implies f \stackrel{H}{\sim} g$
- (10) $f \stackrel{H}{\sim} g \not\implies f \xleftarrow{H} g$

Proof. The only ones that need any proof are the implication failures.

$f \stackrel{H}{\sim} g \not\implies f \xleftarrow{P} g$: If $f = x^2$, $g = 1$ and $t > 0$ then $f + tg = x^2 + t \in \mathcal{H}$, but it is easy to check that $x^2 + y \notin P_2^+$.

$f \stackrel{H}{\sim} g \not\implies f \xleftarrow{H} g$: The above example also shows this, since $x^2 + y \notin \mathcal{H}_2$.

$f \stackrel{P}{\sim} g \not\implies f \xleftarrow{P} g$: If we let $f = x(x+3)$ and $g = (x+1)(x+2)$ then $f \stackrel{P}{\sim} g$. However, we have seen that $f + yg \notin P_2^+$.

□

Fact 11.

- (1) If $\sum f_i(\mathbf{x})y^i \in \mathcal{H}$ then $f_k(\mathbf{x}) \stackrel{H}{\sim} f_{k+2}(\mathbf{x})$ for $k = 0, 1, \dots$.
- (2) If $f(\mathbf{x}) + \dots + yz g(\mathbf{x}) + \dots \in \mathcal{H}$ then $f \stackrel{H}{\sim} g$.
- (3) If $f \xleftarrow{H} g$ then $f \stackrel{H}{\sim} xg$.
- (4) If $f \xleftarrow{H} g$ and $f_1 \xleftarrow{H} g_1$ then $ff_1 \stackrel{H}{\sim} gg_1$.

Proof. Since $y^2 + t \in \mathcal{H}$ for positive t , we know $\partial_y^2 + t$ preserves \mathcal{H} . We first differentiate k times, yielding a polynomial in \mathcal{H} . The constant term of

$$\begin{aligned} (\partial_y^2 + t)[k!f_k(\mathbf{x}) + (k+1)!f_{k+1}(\mathbf{x})y + (1/2)(k+2)!f_{k+2}(\mathbf{x})y^2 + \dots] \\ = [tk!f_k(\mathbf{x}) + (k+2)!f_{k+2}(\mathbf{x})] + y(\dots) \end{aligned}$$

is in \mathcal{H} , which proves the first part. For the second part we multiply by $yz + t$ where $t \geq 0$ and consider the coefficient of yz .

The third part follows from first part and the expansion

$$(f + yg)(xy + 1) = f + y(g + xf) + xgy^2$$

Next, we compute

$$(f + yg)(f_1 + yg_1) = ff_1 + y(fg_1 + f_1g) + y^2gg_1$$

and the conclusion follows as above.

□

5. PROPERTIES OF ONE VARIABLE

In this section we assume $d = 1$. When we restrict ourselves to one variable there are some useful techniques for showing interlacing holds. We let Q_i denote the i 'th quadrant.

Fact 12.

- (1) $f \stackrel{H}{\sim} g$ iff $\frac{f}{g}: Q_1 \rightarrow \mathbb{C} \setminus (-\infty, 0)$.
- (2) $f \stackrel{H}{\leftarrow} g$ iff $\frac{f}{g}: Q_1 \rightarrow \overline{RHP}$.
- (3) $f \stackrel{P}{\leftarrow} g$ iff $\frac{f}{g}: Q_1 \rightarrow Q_1$.
- (4) $f \stackrel{P}{\sim} g$ iff $\frac{f}{g}: Q_1 \rightarrow RHP$
- (5) $f \stackrel{P}{\sim} g$ implies $f \stackrel{H}{\leftarrow} g$.

Proof. $f \stackrel{H}{\sim} g$ iff $f + tg \in \mathcal{H}_1$ for $t > 0$ iff $-\frac{f}{g}(\sigma) \notin (0, \infty)$ for $\sigma \in RHP$ iff $\frac{f}{g}: RHP \rightarrow \mathbb{C} \setminus (-\infty, 0)$ iff $\frac{f}{g}: Q_1 \rightarrow \mathbb{C} \setminus (-\infty, 0)$ since f and g have real coefficients.

Next, $f \stackrel{H}{\leftarrow} g$ iff $f + yg \in \mathcal{H}_2$ iff $-\frac{f}{g}(\sigma) \notin RHP$ for $\chi \in RHP$ iff $\frac{f}{g}: RHP \rightarrow \overline{RHP}$ iff $\frac{f}{g}: Q_1 \rightarrow \overline{RHP}$.

If $f \stackrel{P}{\leftarrow} g$ then we take f to be monic, and we can write $f = \prod (x + r_i)$ and $g = \sum a_i f / (x + r_i)$ where r_i and a_i are non-negative. If $\sigma \in Q_1$ then $(g/f)(\sigma) = \sum a_i / (\sigma + r_i) \in Q_4$, and so $(f/g)(\sigma) \in Q_1$.

If $f + tg \in P_1^+$ for all positive t then f and g have a common interlacing[6], so there is an h satisfying $h \leftarrow f$ and $h \leftarrow g$. If $\sigma \in Q_1$ then $(f/g)(\sigma) = \frac{f}{h} \frac{h}{g}(\sigma)$. By the first part $(h/f)(\sigma) \in Q_1$ and $(g/h)(\sigma) \in Q_4$, so $(f/g)(\sigma) \in RHP$.

The last one follows from the previous ones. □

Fact 13.

- (1) If $f \stackrel{P}{\sim} f_1, \dots, f \stackrel{P}{\sim} f_n$ then $f \stackrel{P}{\sim} f_1 + \dots + f_n$.
- (2) If $f \stackrel{P}{\leftarrow} f_i$ and $g \stackrel{P}{\leftarrow} g_i$ for $1 \leq i \leq n$ then $fg \stackrel{H}{\sim} \sum f_i g_i$.
- (3) If $f \stackrel{P}{\leftarrow} g$ then $\left| \frac{f}{f'} \frac{g}{g'} \right|$ is stable.
- (4) If $f \stackrel{P}{\leftarrow} g$ then the Bezout polynomial $B(x, y) = \frac{1}{x-y} \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix}$ is stable.

Proof. The first one follows from Fact 12(4) since $\frac{f_1 + \dots + f_n}{f} = \frac{f_1}{f} + \dots + \frac{f_n}{f}$. The second follows from

$$\frac{1}{fg} \sum f_i g_i = \sum \frac{f_i}{f} \frac{g_i}{g} \in UHP$$

by Fact 12(3). For the next one, write $f = \prod(x + r_i)$ and $g = \sum a_i \frac{f}{x+r_i}$ where the a_i are non-negative then

$$(f'g - fg')(\sigma) = f^2 \sum \frac{a_i}{(\sigma + r_i)^2}$$

and this is in the lower half plane, so $f^2 \stackrel{H}{\leftarrow} f'g - g'h$. In particular, $f'g - g'h$ is stable.

Finally, we show that $f(x)f(y) \stackrel{H}{\sim} B(x, y)$ by using the identity

$$B(x, y) = \sum a_i \frac{f(x)}{x - r_i} \frac{f(y)}{y - r_i}$$

□

Example 2. Assume that f_0, f_1, f_2, \dots is an orthogonal polynomial sequence. The Christoffel-Darboux formula states that

$$f_0(x)^2 + f_1(x)^2 + \dots + f_n(x)^2 = \frac{k_n}{k_{n+1}} \begin{vmatrix} f_n(x) & f_{n+1}(x) \\ f'_n(x) & f'_{n+1}(x) \end{vmatrix}$$

for certain constants k_n . It follows from Fact 13 that

$$f_0(x)^2 + f_1(x)^2 + \dots + f_n(x)^2 \text{ is stable.}$$

We can generalize the third part of Fact 13 in several ways. The proofs are similar to Fact 13 – write the desired determinant in terms of the parameters and simple geometry implies it satisfies the conditions for positive interlacing.

Fact 14.

- (1) If $\sum f_i(x)y^i \in P_2^+$ and $0 < \alpha < 2$ then $(f_0)^2 \stackrel{H}{\sim} \begin{vmatrix} f_1 & \alpha f_0 \\ f_2 & f_1 \end{vmatrix}$.
- (2) If $\sum f_i(x)y^i \in P_2^+$ then $\begin{vmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{vmatrix}$ is stable.
- (3) Suppose that D_1, D_2, D_3 are positive definite matrices, where D_1 is diagonal, and D_2, D_3 have all positive entries. If $|Id + xD_1 + yD_2 + zD_3| = f(x) + yg(x) + zh(x) + yzk(x) + \dots$ then $\begin{vmatrix} f & g \\ h & k \end{vmatrix}$ is stable.

6. QUESTIONS

Question 1. If $f \stackrel{P}{\sim} g$ or $f \stackrel{H}{\sim} g$ then do f and g have a common interlacing? This is true for $f \stackrel{P}{\sim} g$ where $f, g \in P_1^+$.

Question 2. If $\sum f_{ij}(x)y^i z^j \in P_3^+$ and r is a positive integer then is the following determinant stable?

$$\begin{vmatrix} f_{00} & f_{10} & \cdots & f_{r0} \\ f_{01} & f_{11} & \cdots & f_{r1} \\ \vdots & \vdots & & \vdots \\ f_{0r} & f_{1r} & \cdots & f_{rr} \end{vmatrix}$$

REFERENCES

- [1] Julius Borcea, Petter Brändén, and Boris Shapiro, *Polya-Schur master theorems for circular domains and their boundaries*, available at [arXiv:math/0607416](#).
- [2] Julius Borcea, Petter Brändén, and Boris Shapiro, *Classification of hyperbolicity and stability preservers: the multivariate Weyl algebra case*, available at [arXiv:math.CA/0606360](#).
- [3] P. Brändén, *Polynomials with the half-plane property and matroid theory*, available at [arXiv:math.CO/0605678](#).
- [4] Julius Borcea, Petter Brändén, and Boris Shapiro, *Applications of stable polynomials to mixed determinants: Johnson's conjectures, unimodality and symmetrized Fischer products*, available at [math.SP/0607755](#).
- [5] Julius Borcea, Petter Brändén, and Thomas M. Liggett, *Negative dependence and the geometry of polynomials*, available at [arXiv:0707.2340](#).
- [6] Steve Fisk, *Polynomials, roots, and interlacing*, available at [arXiv:math.CA/0612833](#).
- [7] ———, *An introduction to upper half plane polynomials*. [arXiv:0711.4043](#).
- [8] Jürgen Garloff and David G. Wagner, *Hadamard products of stable polynomials are stable*, Journal of mathematical analysis and applications **202** (1996), 797-809.
- [9] C.R. Johnson, *A characteristic polynomial for matrix pairs*, Linear and multilinear algebra **25** (1989), 289-290.
- [10] Adrian S. Lewis and Pablo A. Parrilo and Motakuri V. Ramana, *The Lax conjecture is true*, available at [arXiv:math.OC/0304104](#).
- [11] Q. I. Rahman and G. Schmeisser, *Analytic Theory of polynomials*, Mathematical Society Monographs, Oxford Science Publications, 2002.
- [12] Vladimir Kharitonov, *Advances in mathematical systems theory* (2001), 19-31. A volume in honor of Diederich Hinrichsen.
- [13] Young-Bin Choe, James G. Oxley, Alan D. Sokal, and David G. Wagner, *Homogeneous multivariate polynomials with the half-plane property*, Adv. in Appl. Math. **32** (2004), no. 1-2, 88-187. Special issue on the Tutte polynomial.